

Universal Divisibility Theorem for Odd Integers

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Introduction

This theorem has been developed for finding the divisibility tests for integers with the aim to easy process.

Statement

For any odd integer 'A' not divisible by five,

- If 'A' divides any number, one more than any multiple of ten, say $A \mid 10x+1$ ($x \in \mathbb{I}$), then any integer is divisible by 'A' if and only if the last digit of the dividend when multiplied by, 'x' increased or decreased by any multiple of 'A' including zero [$x \pm Ak$, ($k \in \mathbb{I}$)], and subtracted from the remaining part of number gives difference divisible by 'A'.
- If 'A' divides any number, one less than any multiple of ten, say $A \mid 10x-1$ ($x \in \mathbb{I}$), then any integer is divisible by 'A' if and only if the last digit of the dividend when multiplied by, 'x' increased or decreased by any multiple of 'A' including zero [$x \pm Ak$, ($k \in \mathbb{I}$)], and added to the remaining part of the number gives the sum divisible by 'A'.

Proof

Proof For i- It is given that 'A' is any odd integer but not divisible by five, 'A' divides any number one more than a multiple of 10, $A \mid 10x+1$, ($x \in \mathbb{I}$)

Let the integer be an '(n+1)-digit' integer $= a_n a_{n-1} a_{n-2} \dots a_0 = 10^n a_n + 10^{n-1} a_{n-1} + 10^{n-2} a_{n-2} + \dots + 10^0 a_0$

and A divides $10^n a_n + 10^{n-1} a_{n-1} + \dots + 10^0 a_0$

$$\Rightarrow A \mid 10^n a_n + 10^{n-1} a_{n-1} + \dots + 10 a_1 + a_0 \quad \dots (i)$$

Now,

$$\begin{aligned} & 10^n a_n + 10^{n-1} a_{n-1} + \dots + 10 a_1 + a_0 \\ &= 10[10^{n-1} a_n + 10^{n-2} a_{n-1} + \dots + a_1] + a_0 \\ &= 10[10^{n-1} a_n + 10^{n-2} a_{n-1} + \dots + a_1] + a_0 + (10x+1) a_0 - (10x+1) a_0 \\ &= 10[10^{n-1} a_n + 10^{n-2} a_{n-1} + \dots + a_1] + a_0 + (10x+1) a_0 - 10x a_0 - a_0 \end{aligned}$$

$$\begin{aligned}
 &= 10[10^{n-1}a_n + 10^{n-2}a_{n-1} + \dots + a_1] - 10xa_0 + (10x+1)a_0 + (a_0 - a_0) \\
 &= 10[10^{n-1}a_n + 10^{n-2}a_{n-1} + \dots + a_1 - xa_0] + (10x+1)a_0 \quad \dots\dots\dots(ii)
 \end{aligned}$$

Using (i) & (ii), we get,

$$A \mid 10[10^{n-1}a_n + 10^{n-2}a_{n-1} + \dots + a_1 - xa_0] + (10x+1)a_0$$

Clearly, $A \mid (10x+1)a_0$ [given that $A \mid (10x+1)$]

Thus, 'A' must divide $10[10^{n-1}a_n + 10^{n-2}a_{n-1} + \dots + a_1 - xa_0]$

$$\Rightarrow A \mid 10[10^{n-1}a_n + 10^{n-2}a_{n-1} + \dots + a_1 - xa_0]$$

Since, A is not a multiple of 2 and 5.

$$\Rightarrow A \text{ does not divide } 10$$

$$\text{Thus, } A \mid 10^{n-1}a_n + 10^{n-2}a_{n-1} + \dots + a_1 - xa_0$$

$$\Rightarrow A \mid a_n a_{n-1} a_{n-2} \dots a_1 - xa_0$$

$$\text{Since, } A \mid Aka_0 (k \in \mathbb{I})$$

$$\Rightarrow A \mid a_n a_{n-1} a_{n-2} \dots a_1 - xa_0 \pm Aka_0 (k \in \mathbb{I})$$

$$\Rightarrow A \mid a_n a_{n-1} a_{n-2} \dots a_1 - (x \pm Ak)a_0$$

Therefore, A divides any integer if and only if the last digit (i.e. a_0) when multiplied by, 'x' increased or decreased by any multiple of 'A' including zero $[x \pm Ak, (k \in \mathbb{I})]$, and subtracted from the remaining part of the number gives difference divisible by 'A'. Where $A \mid 10x+1$

Proof For ii- It is given that 'A' is any odd integer but not divisible by five, A divides any number one less than a multiple of 10,

$$A \mid 10x-1 \quad (x \in \mathbb{I})$$

$$\text{Let the integer be an '(n+1)-digit' integer} = a_n a_{n-1} a_{n-2} \dots a_0 = 10^n a_n + 10^{n-1} a_{n-1} + 10^{n-2} a_{n-2} + \dots + 10^0 a_0$$

$$\text{and } A \text{ divides } 10^n a_n + 10^{n-1} a_{n-1} + \dots + 10^0 a_0$$

$$\Rightarrow A \mid 10^n a_n + 10^{n-1} a_{n-1} + \dots + 10^0 a_0 \quad \dots\dots\dots(i)$$

Now,

$$10^n a_n + 10^{n-1} a_{n-1} + \dots + 10^0 a_0$$

$$= 10[10^{n-1} a_n + 10^{n-2} a_{n-1} + \dots + a_1] + a_0$$

$$\begin{aligned}
 &= 10[10^{n-1}a_n + 10^{n-2}a_{n-1} + \dots + a_1] + a_0 + (10x-1)a_0 - (10x-1)a_0 \\
 &= 10[10^{n-1}a_n + 10^{n-2}a_{n-1} + \dots + a_1] + a_0 - (10x-1)a_0 + 10xa_0 - a_0 \\
 &= 10[10^{n-1}a_n + 10^{n-2}a_{n-1} + \dots + a_1] + 10xa_0 - (10x-1)a_0 + (a_0 - a_0) \\
 &= 10[10^{n-1}a_n + 10^{n-2}a_{n-1} + \dots + a_1 + xa_0] - (10x-1)a_0 \quad \dots\dots\dots(ii)
 \end{aligned}$$

Using (i) & (ii), we get,

$$A \mid 10[10^{n-1}a_n + 10^{n-2}a_{n-1} + \dots + a_1 + xa_0] - (10x-1)a_0$$

Clearly, $A \mid (10x-1)a_0$ [given that $A \mid (10x-1)$]

Thus, 'A' must divide $10[10^{n-1}a_n + 10^{n-2}a_{n-1} + \dots + a_1 + xa_0]$

Since, A is not a multiple of 2 and 5.

$\Rightarrow$ A does not divide 10

$\Rightarrow A \mid 10[10^{n-1}a_n + 10^{n-2}a_{n-1} + \dots + a_1 + xa_0]$

Since, A is not a multiple of 2 and 5.

$\Rightarrow$ A does not divide 10

Thus, $A \mid 10^{n-1}a_n + 10^{n-2}a_{n-1} + \dots + a_1 + xa_0$

$\Rightarrow A \mid a_n a_{n-1} a_{n-2} \dots a_1 + xa_0$

Since, $A \mid Aka_0 (k \in \mathbb{I})$

$\Rightarrow A \mid a_n a_{n-1} a_{n-2} \dots a_1 + xa_0 \pm Aka_0 (k \in \mathbb{I})$

$\Rightarrow A \mid a_n a_{n-1} a_{n-2} \dots a_1 + (x \pm Ak)a_0$

Therefore, A divides any integer if and only if the last digit (i.e. a_0) when multiplied by 'x' increased or decreased by any multiple of 'A' including zero [$x \pm Ak, (k \in \mathbb{I})$], and added to the remaining part of the number gives sum divisible by 'A'. Where $A \mid 10x-1$

EXAMPLE 1- Let us find the divisibility of 7 using the theorem.

Clearly, $7 \mid 21$, $(21=10 \cdot 2+1)$

In accordance with STATEMENT-1, $[7 \mid 10x + 1], x=2$

Therefore, any integer is divisible by 7 if and only if the last digit of the integer when multiplied by 2 and subtracted from the remaining part of the number gives difference divisible by 7.

CHECK: for the integer 3437

$$343-2\cdot 7=329, \quad 32-9\cdot 2=14$$

Clearly, $7 \mid 14$

$$\Rightarrow 7 \mid 3437$$

Thus, 3437 is divisible by 7.

Further, in accordance with STATEMENT-1, by increasing x with a multiple of 7, we get,

$$x+7k = 2+7k \quad (k \in \mathbb{I})$$

let ' k ' be equal to -1

$$\Rightarrow 2+7k = 2-7 = -5$$

Therefore, any integer is divisible by 7 if and only if the last digit of the integer when multiplied by -5 and subtracted from the remaining part of the number gives difference divisible by 7 or if the last digit of the integer when multiplied by 5 and added to the remaining part of the number gives sum divisible by 7.

CHECK: for the integer 3437

$$343+5\cdot 7=378, \quad 37+5\cdot 8=77$$

Clearly, $7 \mid 77$

$$\Rightarrow 7 \mid 3437$$

Thus, 3437 is divisible by 7.

EXAMPLE 2- Let us find the divisibility of 13 using the theorem.

Clearly, $13 \mid 39$, ($13=10\cdot 4 - 1$)

In accordance with STATEMENT-2, [$13 \mid 10x - 1$], $x=4$

Therefore, any integer is divisible by 13 if and only if the last digit of the integer when multiplied by 4 and added to the remaining part of the number gives sum divisible by 13.

CHECK: for the integer 5967

$$596+4\cdot 7=624, \quad 62+4\cdot 4=78, \quad 7+4\cdot 8=39$$

Clearly, $13 \mid 39$

$$\Rightarrow 13 \mid 5967$$

Thus, 5967 is divisible by 13.

Further, in accordance with STATEMENT-2, by increasing x with a multiple of 13, we get,

$$x+7k = 4+13k \quad (k \in \mathbb{I})$$

let ' k ' be equal to -1

$$\Rightarrow 4+13k = 4-13 = -9$$

Therefore, any integer is divisible by 13 if and only if the last digit of the integer when multiplied by -9 and added to the remaining part of the number gives sum divisible by 13 or if the last digit of the integer when multiplied by 9 and subtracted from the remaining part of the number gives difference divisible by 13.

CHECK: for the integer 5967

$$596-9 \cdot 7=533, \quad 53-9 \cdot 3=13$$

Clearly, $13 \mid 13$

Thus, 5967 is divisible by 13.

$$\Rightarrow 13 \mid 5967$$

EXAMPLE 3- Let us find the divisibility of 11 using the theorem.

$$\text{Clearly, } 7 \mid 11, \quad (11=10 \cdot 1+1)$$

In accordance with STATEMENT-1, $[7 \mid 10x + 1], x=1$

Therefore, any integer is divisible by 11 if and only if the last digit of the integer when multiplied by 1 and subtracted from the remaining part of the number gives difference divisible by 7 or if the last digit of the integer when subtracted from the remaining part of the number gives difference divisible by 11.

CHECK: for the integer 47036

$$4703-6=4697, \quad 469-7=462, \quad 46-2=44,$$

Clearly, $11 \mid 44$

$$\Rightarrow 11 \mid 47036$$

Thus, 47036 is divisible by 11.

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References

1. Excursion in Mathematics.
2. Pre-College Mathematics.